

Energy spectrum of the Manning-Rosen potential including centrifugal term solved by exact and proper quantization rules

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Abstract The energy spectrum of the Manning-Rosen potential including centrifugal term in higher dimensions is presented by exact quantization rule approach. The result is compared with that by proper quantization rule method. It is found that the latter is better than that of the exact quantization rule. We find that the interdimensional degeneracy exists for the states in different dimensions. For the special case $D = 3$, the results agree well with those obtained by other methods.

Keywords Exact and proper quantization rules · Manning-Rosen potential · Centrifugal term · Interdimensional degeneracy

1 Introduction

The exact solutions of solvable quantum systems have received much attention [1–3]. Among various solvable potentials [4, 5], the Manning-Rosen potential introduced for describing diatomic molecular vibrations [6] has been widely applied to molecular physics and quantum chemistry. Its bound states and scattering properties have been investigated by a variety of techniques [7–12].

Recently, an exact quantization rule approach [13, 14] has been proposed and a great deal of important physical problems [15–18] have been reexamined through this

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method. This methodology has been shown to be effective for calculating the bound state energy levels of the Schrödinger equation with a spherical potential. Nevertheless, it should be noted that the integrals involved in exact quantization rule, in particular the calculation of the quantum correction term become rather difficult. To improve this method, we have found a proper quantization rule [19], which takes advantage over previous exact quantization rule.

The purpose of this work is two-fold. The first is to deal with the Manning-Rosen potential including the centrifugal term in higher dimensions via the exact quantization rule method. The second is to treat it by the proper quantization rule and then compare with them.

This work is organized as follows. In Sect. 2 we briefly review the exact quantization rule method and its extension to arbitrary dimensions. In Sect. 3 we apply this method to obtain the energy spectrum for the Manning-Rosen potential. The result through the proper quantization rule shall be presented in Sect. 4. The interdimensional degeneracies among states are discussed in Sect. 5. Finally, some conclusions are given in Sect. 6.

2 Exact quantization rule

We begin by a brief review of exact quantization rule method [13]. It is known that the one-dimensional Schrödinger equation $\psi''(x) = -2\mu/\hbar^2[E - V(x)]\psi(x)$ is equivalent to a non-linear Riccati equation

$$-\phi'(x) = \frac{2\mu}{\hbar^2}[E - V(x)] + \phi(x)^2, \quad (1)$$

where $\phi(x) = \psi'(x)/\psi(x)$ is the logarithmic derivative of wave function $\psi(x)$. The exact quantization rule was proposed for one-dimensional Schrödinger equation as

$$\int_{x_A}^{x_B} k(x)dx = N\pi + \int_{x_A}^{x_B} k'(x) \frac{\phi(x)}{\phi'(x)} dx, \quad (2)$$

$$k(x) = \sqrt{2\mu[E - V(x)]}/\hbar, \quad E \geq V(x), \quad (3)$$

where x_A and x_B are two turning points determined by $E = V(x)$. $N = n + 1$ is the number of nodes of $\phi(x)$ in the region $E \geq V(x)$ and it is larger by one than the number n of nodes of wave function $\psi(x)$. The first term $N\pi$ is the contribution from the nodes of the logarithmic derivative of wave function, and the second is called the quantum correction. It is found that, for all well-known exactly solvable quantum systems, this quantum correction is independent of the number of nodes of wave function. Therefore, it is enough to consider the ground state in calculating the quantum correction $Q_0 = \int_{x_A}^{x_B} k'_0(x)\phi_0(x)/\phi'_0(x)dx$. This exact quantization rule was also generalized to three and arbitrary dimensional cases [13, 17]. The exact quantization rule in arbitrary dimensions was written as [17]

$$\int_{r_A}^{r_B} k(r)dr = N\pi + \int_{r_A}^{r_B} k'(r) \frac{\phi(r)}{\phi'(r)} dr, \tag{4}$$

$$k(r) = \sqrt{2\mu[E - V_{\text{eff.}}(r)]/\hbar}, \quad E \geq V_{\text{eff.}}(r), \tag{5}$$

where

$$V_{\text{eff.}}(r) = V(r) + \frac{\ell'(\ell' + 1)}{r^2}, \quad \ell' = \ell + \frac{D - 3}{2}. \tag{6}$$

As what follows, we shall first employ this method to solve the D -dimensional Schrödinger equation for Manning-Rosen potential and then solve it via proper quantization rule method.

3 Energy spectrum

In three dimensional space the Manning-Rosen effective potential is defined as [6]

$$V_{\text{eff.}}(r) = \frac{1}{q^2} \left[\frac{\beta(\beta - 1)\alpha^2 e^{-2\alpha r}}{(1 - e^{-\alpha r})^2} - \frac{A\alpha^2 e^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{\ell(\ell + 1)}{r^2} \right], \quad q^2 = 2\mu/\hbar^2. \tag{7}$$

It is known that this quantum system can be solved analytically only for $\ell = 0$ case [9]. Nevertheless, by taking a suitable approximation $1/r^2 \simeq \alpha^2 e^{-\alpha r} / (1 - e^{-\alpha r})^2$ to centrifugal term [10,20], approximately analytical solution to this system for arbitrary ℓ -state was studied [10]. One may naturally express the Manning-Rosen effective potential in D dimensions as follows:

$$V_{\text{eff.}}(r) = \frac{\alpha^2}{q^2} \left[\frac{\beta(\beta - 1)e^{-2\alpha r}}{(1 - e^{-\alpha r})^2} - \frac{Ae^{-\alpha r}}{1 - e^{-\alpha r}} + \frac{\ell'(\ell' + 1)e^{-\alpha r}}{(1 - e^{-\alpha r})^2} \right], \tag{8}$$

where ℓ' is defined in Eq. (6). By taking the wave functions $\psi(\mathbf{r}) = r^{-(D-1)/2} R(r) Y_{l_D-2, \dots, l_1}^l(\mathbf{x})$ and substituting it into the Schrödinger equation in arbitrary dimensions [21,22], the effective Manning-Rosen potential shall reduce to those special cases for $D = 3, 2, 1$. For example, when $D = 3$, one has $\ell' = \ell$ as usual three-dimensional case. The centrifugal term becomes $\ell(\ell + 1)/r^2$; when $D = 2$, the centrifugal term becomes $(m^2 - 1/4)$ where $\ell = m$ in two dimensions; when $D = 1$, we have $\ell = 0$, thus the centrifugal term disappears totally.

In the following, we shall apply the exact quantization rule to find its energy spectrum. By introducing a new variable y

$$y = \frac{1}{e^{\alpha r} - 1}, \quad y'(r) = -\alpha y(1 + y), \quad y \in (\infty, 0), \quad r \in (0, \infty), \tag{9}$$

eq. (8) is transformed to the form

$$V_{\text{eff.}}(r) = \frac{\alpha^2}{q^2} \left\{ [\beta(\beta - 1) + \ell'(\ell' + 1)]y^2 + [\ell'(\ell' + 1) - A]y \right\}. \quad (10)$$

To show a unified way for dealing with such potentials, we denote the coefficients in the effective potential by two parameters V_2 and V_1 , respectively, i.e.,

$$\begin{aligned} V_2 &= \alpha^2 \delta / q^2, & V_1 &= \alpha^2 \gamma / q^2, & V(r) &= V_2 y^2 + V_1 y, \\ \delta &= [\beta(\beta - 1) + \ell'(\ell' + 1)], & \gamma &= [\ell'(\ell' + 1) - A]. \end{aligned} \quad (11)$$

Let $y_A = 1/(e^{\alpha r_A} - 1)$ and $y_B = 1/(e^{\alpha r_B} - 1)$, where r_A and r_B are two turning points satisfying $V(r_A) = V(r_B) = E_{nl}$. One finds

$$y_A = -\frac{V_1 + \sqrt{V_1^2 + 4E_{nl}V_2}}{2V_2}, \quad y_B = \frac{-V_1 + \sqrt{V_1^2 + 4E_{nl}V_2}}{2V_2}, \quad (12)$$

with the following properties

$$y_A + y_B = -\frac{V_1}{V_2}, \quad y_A y_B = -\frac{E_{nl}}{V_2}. \quad (13)$$

The momentum $k(r)$ and its first derivative between two turning points can then be written as

$$k(y) = \sqrt{2\mu V_2(y - y_A)(y_B - y)} / \hbar, \quad k'(y) = \frac{\sqrt{2\mu V_2}}{2\hbar} \left(\sqrt{\frac{y_B - y}{y - y_A}} - \sqrt{\frac{y - y_A}{y_B - y}} \right). \quad (14)$$

On the other hand, the Riccati Eq. (1) becomes

$$\alpha y(1 + y)\phi_0'(y) = \frac{2\mu}{\hbar^2} [E_0 - V_2 y^2 - V_1 y] + \phi_0(y)^2. \quad (15)$$

The solutions of Eq. (15) suggest for the logarithmic derivative $\phi_0(y)$ the following expression $\phi_0(y) = c_1 y + c_2$. Substitution of this into Eq. (15) allows us to obtain

$$2c_1 c_2 - \frac{2\mu}{\hbar^2} V_1 = \alpha c_1, \quad c_1^2 - \frac{2\mu}{\hbar^2} V_2 = \alpha c_1, \quad \frac{2\mu}{\hbar^2} E_0 = -c_2^2. \quad (16)$$

The solutions are given by

$$c_2 = \frac{\alpha}{2} + \frac{\mu V_1}{\hbar^2 c_1}, \quad c_1 = m\alpha, \quad (17)$$

where

$$m = \frac{1}{2} + \sqrt{\delta + \frac{1}{4}}, \quad \delta = m(m - 1). \tag{18}$$

It should be pointed out that we only choose the positive sign in front of the square root for m . This is because the wave function $\psi(r)$ obtained from logarithmic derivative $\phi_0(r) = \frac{1}{\psi(r)} \frac{d\psi(r)}{dr} = \frac{1}{\psi(r)} \frac{dy}{dr} \frac{d\psi(r)}{dy} = c_1 y + c_2$ with $c_1 = m\alpha$ will decrease exponentially, which is required physically. On the other hand, it is shown from Eq. (15) that c_1 has to be positive since $2\mu/\hbar^2 [E_0 - V_2 y^2 - V_1 y] + \phi_0(y)^2$ on the right hand side of Eq. (15) is positive for $E > V_{\text{eff}}$, as shown in Eqs. (4) and (5). Thus, one has

$$\phi_0(y) = m\alpha y + \frac{\alpha}{2} + \frac{\mu V_1}{m\alpha\hbar^2}, \quad E_0 = -\frac{\hbar^2}{2\mu} \left(\frac{\alpha}{2} + \frac{\mu V_1}{m\alpha\hbar^2} \right)^2. \tag{19}$$

In order to calculate quantum correction Q_0 and the integral (4) of the momentum $k(r)$, we shall make use of the following helpful integral formulas

$$\int_{r_A}^{r_B} \frac{dr}{\sqrt{(r_B - r)(r - r_A)}} = \pi, \tag{20}$$

$$\int_{r_A}^{r_B} \frac{1}{r} \sqrt{(r - r_A)(r_B - r)} dr = \frac{\pi}{2} (r_A + r_B) - \pi \sqrt{r_A r_B}, \tag{21}$$

$$\int_{r_A}^{r_B} \frac{dr}{(a + br)\sqrt{(r_B - r)(r - r_A)}} = \frac{\pi}{\sqrt{(a + br_B)(a + br_A)}}. \tag{22}$$

Now, the quantum correction is calculated from the ground state

$$\begin{aligned} \int_{r_A}^{r_B} k'_0(r) \frac{\phi_0(r)}{\phi'_0(r)} dr &= -\frac{\sqrt{2\mu V_2}}{2\hbar\alpha} \int_{y_A}^{y_B} \frac{y + c_2/c_1}{y(1 + y)} \left(\sqrt{\frac{y_B - y}{y - y_A}} - \sqrt{\frac{y - y_A}{y_B - y}} \right) dy \\ &= \frac{\sqrt{2\mu V_2}}{\hbar\alpha} \int_{y_A}^{y_B} \frac{(y + c_2/c_1)(y - (y_A + y_B)/2)}{y(1 + y)\sqrt{(y - y_A)(y_B - y)}} dy \\ &= \frac{\sqrt{2\mu V_2}}{\hbar\alpha} \int_{y_A}^{y_B} \left[1 + \frac{c_2 V_1}{2c_1 V_2 y} + \frac{(c_2 - c_1)(2V_2 - V_1)}{(2c_1 V_2)(1 + y)} \right] \\ &\quad \times \frac{1}{\sqrt{(y - y_A)(y_B - y)}} dy \end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2\mu V_2}}{\hbar\alpha} \pi \left[1 + \frac{\sqrt{2\mu} V_1}{2c_1 \hbar \sqrt{V_2}} + \frac{\sqrt{2\mu}(2V_2 - V_1)}{2c_1 \hbar \sqrt{V_2}} \right] \\
&= \frac{\sqrt{2\mu V_2}}{\hbar\alpha} \pi \left(1 + \frac{\sqrt{2\mu} V_2}{c_1 \hbar} \right) \\
&= \pi \left(\sqrt{\delta} + m - 1 \right). \tag{23}
\end{aligned}$$

the integral of the momentum $k(r)$ gives

$$\begin{aligned}
\int_{r_A}^{r_B} k(r) dr &= \int_{y_A}^{y_B} \frac{\sqrt{2\mu V_2(y - y_A)(y_B - y)}/\hbar}{-\alpha y(1 + y)} dy \\
&= -\frac{\sqrt{2\mu V_2}}{\hbar\alpha} \int_{y_A}^{y_B} \left(\frac{\sqrt{(y - y_A)(y_B - y)}}{y} - \frac{\sqrt{(y - y_A)(y_B - y)}}{1 + y} \right) dy \\
&= -\pi \sqrt{\delta} \left(\sqrt{\frac{-V_1}{V_2} - \frac{E_{n\ell}}{V_2} + 1} - \sqrt{\frac{-E_{n\ell}}{V_2} - 1} \right). \tag{24}
\end{aligned}$$

By successively using the relation in Eqs. (23) and (24), the exact quantization rule (4) can be expressed as

$$-\pi \sqrt{\delta} \left(\sqrt{\frac{-V_1}{V_2} - \frac{E_{n\ell}}{V_2} + 1} - \sqrt{\frac{-E_{n\ell}}{V_2} - 1} \right) = (n + 1)\pi + \pi(\sqrt{\delta} + m - 1). \tag{25}$$

Hence, one can immediately obtain the energy spectrum for the D -dimensional Manning-Rosen potential $E_{n\ell}$ as

$$E_{n\ell} = -\frac{\hbar^2 \alpha^2}{8\mu} \left[\tau - \frac{A + \beta(\beta - 1)}{\tau} \right]^2, \quad \tau = n + m, \tag{26}$$

where the relations $\delta - \gamma = A + \beta(\beta - 1)$ and $\delta = m(m - 1)$ are used.

We find that Eq. (26) is consistent with the energy levels

$$E_{n\ell} = -\frac{\hbar^2}{32\mu b^2} \left[\frac{4(n + 1)^2 + (D + 2\ell - 2)^2 + 4(2n + 1)\eta - 4A - 1}{2(n + 1 + \eta)} \right]^2 \tag{27}$$

presented in Ref. [12] noting that $1/b = \alpha$, $\eta = m - 1$.

We now discuss the special case $D = 3$. If so, Eq. (26) coincides with the result

$$E_{n\ell} = -\frac{1}{2b^2} \left[\frac{(n + 1)^2 - A + \ell(\ell + 1) + (2n + 1)\delta}{2(n + \delta + 1)} \right]^2 \tag{28}$$

proposed in Ref. [10] noting that $1/b = \alpha, \delta = m - 1$. When $D = 3$ and $\ell = 0$, it provides the same result as

$$E_v = -\frac{1}{k\rho^2} \left[\frac{A - \beta}{2(\beta + v)} - \frac{v(v + 2\beta)}{2(\beta + v)} \right]^2 \tag{29}$$

given in Ref. [6,9] noting that $1/\rho = \alpha, v = n$, and $k = 2\mu/\hbar^2$.

In particular, when $D = 3, \beta = 0$ or $1, A = 2\mu Ze^2/\hbar^2\alpha$, the Manning-Rosen potential is reduced to Hulthén potential. Meanwhile, Eq. (26) turns out to be

$$E_{n\ell} = -\frac{\hbar^2}{2\mu} \left[\frac{\mu Ze^2}{\hbar^2(n + \ell + 1)} - \frac{(n + \ell + 1)\alpha}{2} \right]^2, \tag{30}$$

which obviously agrees well with that given in Ref. [23].

4 Proper quantization rule

In this section, we are going to make use of recently proposed proper quantization rule to treat this potential again. To this end, let us first review this proper quantization rule [19], which is given by

$$\int_{x_A}^{x_B} k(x)dx - \int_{x_{0A}}^{x_{0B}} k_0(x)dx = (N - 1)\pi = n\pi \tag{31}$$

for one dimensional case and

$$\int_{r_A}^{r_B} k(r)dr - \int_{r_{0A}}^{r_{0B}} k_0(r)dr = n\pi. \tag{32}$$

for higher dimensional case. The two integrals involved in the proper quantization rule have the same mathematical form. Accordingly, when applying it to calculate the energy levels we can calculate its first integral with respect to $k(x)$ or $k(r)$, and then replace energy levels E_n in the result by the ground state energy E_0 to obtain the second integral. This will greatly simplify the complicated calculations encountered previously. As an illustration, we apply this proper quantization rule to study this potential again in order to show its great advantage. As known above, our aim is to find the energy level of the ground state, which has been given in Eq. (19) as

$$E_0 = -\frac{\hbar^2\alpha^2}{8\mu} \left(1 + \frac{\gamma}{m} \right)^2, \tag{33}$$

by considering $\gamma = 2\mu V_1/\alpha^2\hbar^2$ since it is enough to know the eigenvalue of the ground state by the proper quantization rule.

The integral of the momentum $k(r)$ has been calculated in Eq. (24) as

$$\int_{r_A}^{r_B} k(r) dr = -\pi\sqrt{\delta} \left(\sqrt{\frac{-V_1}{V_2} - \frac{E_{n\ell}}{V_2} + 1} - \sqrt{\frac{-E_{n\ell}}{V_2} - 1} \right). \quad (34)$$

By using Eq. (32) we have

$$\begin{aligned} & -\pi\sqrt{\delta} \left(\sqrt{\frac{-V_1}{V_2} - \frac{E_{n\ell}}{V_2} + 1} - \sqrt{\frac{-E_{n\ell}}{V_2} - 1} \right) \\ & + \left[\pi\sqrt{\delta} \left(\sqrt{\frac{-V_1}{V_2} - \frac{E_0}{V_2} + 1} - \sqrt{\frac{-E_0}{V_2} - 1} \right) \right] = n\pi. \end{aligned} \quad (35)$$

One can thus immediately obtain the eigenvalues for the D -dimensional Manning-Rosen potential $E_{n\ell}$ as Eq. (26), where the relations $\delta - \gamma = A + \beta(\beta - 1)$ and $\delta = m(m - 1)$ are used.

Before ending this part, we are going to give a useful remark on Eq. (26). When $D = 1$, this result is the same as those of [6,9] and as Eq. (30) of [24], in which the tridiagonal program was used by Zhang et al. except for a slight different factor 2 in front of the $\beta(\beta - 1)$. Essentially they are same since δ depends on the parameter $q^2 = 2\mu/\hbar^2$.

5 Interdimensional degeneracies

Let us consider the interdimensional degeneracies of this system. To our knowledge, almost all quantum systems in higher dimensions exist such a property as shown in Refs. [25–29] and also our recent work [30,31]. This is a common phenomenon in higher dimensional wave equations. We attempt to reprove the existence of this property for this system.

Since we have derived the explicit energy expression from the Schrödinger equation, it becomes possible to study the complete spectrum of interdimensional degeneracies for a Manning-Rosen potential system. By using Eqs. (11) and (18), Eq. (26) may also be written as

$$\begin{aligned} E_{n\ell} = & -\frac{\hbar^2\alpha^2}{8\mu} \left[\frac{\beta(\beta - 1) + A}{n + 1/2 + \sqrt{(\beta - 1/2)^2 + (\ell + \frac{D-1}{2})(\ell + \frac{D-3}{2})}} \right. \\ & \left. - \left(n + 1/2 + \sqrt{(\beta - 1/2)^2 + (\ell + \frac{D-1}{2})(\ell + \frac{D-3}{2})} \right) \right]^2. \end{aligned} \quad (36)$$

It is evident that Eq. (36) keeps invariant under the interchange

$$\ell, D \iff \ell + i, D - 2i, \quad i = 0, 1, 2, 3, \dots, \quad (37)$$

which means that an isomorphism exists for the Manning-Rosen potential in D dimensions between angular momentum ℓ and dimension D such that each unit increment in ℓ is equivalent to two-unit increment in D [25, 26]. That is to say, for this quantum system the states related by the dimensional link in Eq. (37) are exactly degenerate. This is the so-called interdimensional degeneracy as discussed in [27–29]. Here, we explicitly show its existence again for D dimensional Manning-Rosen potential. This kind of property is like the state degeneracy of hydrogen atom and harmonic oscillator, where the different states correspond to same energy.

6 Conclusions

The energy spectrum of the Manning-Rosen potential including centrifugal term in higher dimensions has been carried out by the exact quantization rule approach and the proper quantization rule, respectively. Also, we have found that the interdimensional degeneracy exists for the states in different dimensions.

Before concluding this work, we summarize its advantages over the original quantization rule. First, in applying the original exact quantization rule, one must find the eigenvalue and eigenfunction of the ground state simultaneously to calculate the complicated quantum correction term, while with the proper quantization rule only the ground state energy is sufficient to determine the energy levels of a quantum system. Second, by using the proper quantization rule we only need to calculate one of the two integrals. Finally, the most important point is that the expression of the proper quantization rule is more symmetrical than the original one. The beauty and simplicity of the rule come from its meaning—whenever the number of the nodes of $\phi(x)$ or the number of the nodes of the wave function $\psi(x)$ increases by one, the momentum integral $\int_{x_A}^{x_B} k(x)dx$ will increase by π . It should be pointed out that even the Manning-Rosen potential in arbitrary dimensions has been treated in [19] by proper quantization rule, the exact quantization rule approach was not illustrated there. This will make the reader further recognize that the proper quantization rule shows more simplicity and symmetry than original exact quantization rule [13].

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